Open Problems on the Validity of Grassmann's Laws

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Hermann Grassmann disclosed rules for assessing colour sameness more than a century ago, and all of basic colour theory (including the CIE standard observers) depends on these rules:

Symmetry: $$\textit{If } \mathbf{A} = \mathbf{B} \textit{ then } \mathbf{B} = \mathbf{A} \qquad (1)$$

Transitivity: $$\textit{If } \mathbf{A} = \mathbf{B} \textit{ and } \mathbf{B} = \mathbf{C} \textit{ then } \mathbf{A} = \mathbf{C} \qquad (2)$$

Proportionality: $$\textit{If } \mathbf{A} = \mathbf{B} \textit{ then } k\mathbf{A} = k\mathbf{B} \qquad (3)$$

Additivity: $$\textit{If } \mathbf{A} = \mathbf{B} \textit{ and } \mathbf{C} = \mathbf{D} \textit{ then } \mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{D} \qquad (4)$$

$$\textit{If } \mathbf{A} = \mathbf{B} \textit{ and } \mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{D} \textit{ then } \mathbf{C} = \mathbf{D} \qquad (5)$$

Grassmann's laws are tested by what is called a *symmetric-matching* experiment: An observer compares two lights that are presented on identical backgrounds and with a visual system adapted the same for both sides of the match.

The laws are known not to be exactly true in human colour matching. Besides the three cone types that herald the trichromacy of vision at high (photopic) light intensities, a fourth photoreceptor type (rods) contributes to vision at low (mesopic and scotopic) light intensities and away from the centre of vision (fovea). At very high light intensities, unbleached photopigments deplete and, in aggregate, change their action spectrum. At still higher light intensities, a photopigment molecule can absorb multiple photons but respond as if it absorbed only one photon. These effects compromise Grassmann's laws, but the successful application of the laws, e.g., in photography and television, has led us to believe that the compromises are not serious.

In 1980, Wyszecki and Stiles published a detailed study of the pigment-bleaching hypothesis, comparing Maxwell-type matches at retinal illuminances of 1000 and 100 000 td. They found strong and predictable bleaching characteristics for the "red" and "green" fundamentals but the "blue" fundamental exhibited unexpected and unexplained behaviour. In 1982, the same authors published colour-matching functions for a single observer measured by the Maxwell and maximum-saturation methods. The deviations are considerable and the authors conclude that they represent failures of the additivity law.

Further cause for questioning the practical sufficiency of Grassmann's laws emerged in 1992, when Thornton conducted symmetric colour-matching experiments to test the transformability of primaries. Through these experiments, Thornton inferred colour-matching functions for six observers using three different sets of nearly monochromatic primary lights, and also for a virtual seventh observer whose colour-matching functions are averaged from the other six observers. His observers made many matches, but each observer made each match only once.

Thornton found, for each observer, that a colour match of a test light with a mixture of three primary lights becomes a substantial mismatch when each of the primaries in this set is replaced by a matching combination of a second set of primaries. Such *transformation of primaries* amounts to two applications of Grassmann's additivity law. (Find the Set-2 match of each primary in Set 1, replace each Set-1 primary with its Set-2 match, and thereby predict the matches made with Set 2 in a new experiment.) Hence Grassmann's laws fail if transformability fails.

The possibility of such failure leads one to ask what generalization could replace Grassmann's laws. Implicit in Grassmann's additivity rule is the interpretation that "+" means addition at each wavelength of light intensities (or quantum fluxes) per unit wavelength interval. Other additivity domains could be imagined. Brill[15,16] tried two theories, each of which contained a parameter whose value could optionally be set to retrieve conventional Grassmann additivity, but covered other alternatives for other parameter values. One such *covering theory* posited that photon counts in given-sized wavelength-time bins undergo a power-function transformation before being summed into three "tristimulus-like" numbers. The other theory posited a depleted optical density of photopigments under more intense lights.

Another question is to what extent statistical variations could account for Thornton's primary-transformation data. In the absence of replicate matches, a numerical simulation was performed about five years ago. The simulation transforms the CIE 1931 CMFs to Thornton's Prime-Color (PC) and Anti-Prime (AP) primaries, adds Gaussian noise to each set of CMFs, and transforms from each set to estimate the CMFs of the other. A Monte-Carlo approach was chosen, because the alternative partial-derivative approach fails if the relative errors are large—i.e., when the 2% of maximum is added to a small "true" value.

The basic finding of the study was that Thornton's observed failures of transformability are consistent with random intra-observer matching noise. This does not *prove* that Thornton's result is a statistical artefact. It merely opens that explanation as a possibility. *The replicate-match experiment is still needed to answer the fundamental question.*

Thornton's findings were discussed at a CIE Symposium on Improved Colorimetry in June 1993. However, the questions remained unresolved. Then, in Warsaw in 1999, CIE Division 1 sought to bring the matter to closure by forming a new technical committee, CIE TC 1-56, "Improved Colour Matching Functions."

The original plan of CIE TC 1-56 had several steps: The first step was to resolve the transformability problem by conducting an experiment with many replicate colour matches for individual observers. The next step was to look at differences between observers, and to weigh the statistical significance of the deviations of the average of these observers from the CIE functions. Finally, if improvements could be made, the committee was chartered to suggest improved colour-matching functions.

After this fairly stringent plan, no experiments were reported until 2005, when three groups indicated that they were doing relevant work: Ronnier Luo, Boris Oicherman, et al. (University of Leeds, UK) at about 3 cd/m^2, Claudio Oleari at about 30 cd/m^2, and Yasuhisa Nakano at about 300 cd/m^2. These expressions of interest solidified at the 2005 meeting of TC1-56 in León, Spain, and all three groups presented their preliminary results. At that point, TC1-56 down-scoped its goals: Henceforth, the main goal is not to find better colour-matching functions, but to test the transformability of primaries for many trials on a single observer.

Within the next four years, the three laboratories above should have gathered enough data so that CIE TC1-56 can assess the usability of Grassmann's laws in the ever-more-demanding environments of today's world.