Intransitive colour matching and metamerism

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Human colour vision starts from processing light by three colour mechanisms (presumably, the cone photoreceptors). The spectral sensitivity of these mechanisms (cone fundamentals) is supposed to be derived from colour matching data. Specifically, the cone fundamentals are assumed to be a linear transformation of the colour matching functions (Smith & Pokorny, 2003). This linear relationship results from Grassmann's laws which are widely believed to take place for colour matching (Wyszecki & Stiles, 1982, p. 118). Two most important of these are

Law of transitivity:

$$\text{for any lights } a, b, \text{ and } c, \text{ if } a \text{ matches } b \text{ and } b \text{ matches } c, \text{ then } a \text{ matches } c. \quad (1)$$

Law of additivity:

$$\text{for any lights } a, b, \text{ and } c, \; a \text{ matches } b \text{ if and only if } (a + c) \text{ matches } (b + c). \quad (2)$$

Lights a and b are believed to match each other (i.e., to be subjectively indistinguishable) if, and only if, the response of each of the colour mechanisms to the light a equals that to light b. However, there is every indication that colour match does not entail the equality of the colour mechanisms' responses. There are some lights that produce different cone responses, and yet a human observer cannot distinguish between them. For example, two monochromatic lights with close wavelengths. Such lights will match each other despite that the colour mechanisms' responses to them are different.

It follows that colour matching is not transitive. Indeed, in a series of monochromatic lights, $\delta(\lambda), \delta(\lambda + \Delta\lambda), \delta(\lambda + 2\Delta\lambda), \ldots, \delta(\lambda + n\Delta\lambda)$, each adjacent pair $\delta(\lambda + i\Delta\lambda)$ and $\delta(\lambda + (i+1)\Delta\lambda)$ may match each other provided $\Delta\lambda$ is small enough. However, the pair $\delta(\lambda)$ and $\delta(\lambda + n\Delta\lambda)$ will obviously be well discriminable (i.e., mismatch each other) for sufficiently large n. While the intransitivity of human colour matching judgements is often assumed to be due to "piling up of small errors" which "must be eliminated by making matches with sufficient statistical precision" (Krantz, 1975, p. 289), it can not be reduced to statistical inference, not to mention ignored.

If one defines metamerism as the equality of the colour mechanisms' responses (i.e., lights a and b are metameric if and only if the response of each colour mechanism to light a equals that to the light b), then metamerism implies colour matching but the converse is not true. This may happen when an observer judges that lights a and b match each other when the response of each colour mechanism to light a is close enough (but not necessarily equal) to that to light b. In the latter case colour matching will be reflexive and symmetric but not transitive.

Moreover, failure to meet the transitivity law (1) implies that of the additivity law (2). Specifically, we prove that if (2) holds for a reflexive and symmetric relation σ then it is transitive. This is in line with those who claimed invalidity of the additivity law on experimental grounds (Thornton, 1992b, 1992a).

In this report we show how metamerism can be defined in terms of colour matching (when the latter is not transitive) so as to satisfy Grassmann's law.

Let A be a set of all lights considered as visual stimuli. Let σ be a *colour matching relation* on A. Thus, $x\sigma y$ stands for "x and y match each other". We assume that σ is a reflexive (for each a $a\sigma a$) and symmetric ($a\sigma b$ implies $b\sigma a$) binary relation on A. Let us define a binary relation $\sim_\sigma$ on A (called *matching metamerism*) as follows. For each a and b in A

$$a \sim_\sigma b \text{ if, and only if, for each } c \in A \ a\sigma c \Leftrightarrow b\sigma c. \tag{3}$$

Therefore, two lights, a and b, are metameric if, and only if, they can substitute each other without breaking matching relation with any light c.

We establish laws which intransitive colour matching σ should follow so that the metamerism $\sim_\sigma$ satisfies Grassmann's laws. Firstly, we prove that colour matching metamerism $\sim_\sigma$ is an equivalence relation providing σ is reflexive and symmetric. Secondly, denote $\sigma(x)$ the set of all lights which match x, that is, $\sigma(x) = \{y \in A : x\sigma y\}$. We prove that $\sim_\sigma$ meets the additivity law (2), and Proportionality law (Wyszecki & Stiles, 1982, p. 118), if the following two axioms are satisfied:

Law of translatory invariance

$$\text{for any lights } a, b, \text{ and } c, \ \sigma(a) = \sigma(b) \text{ if and only if } \sigma(a+c) = \sigma(b+c). \tag{4}$$

Law of positive homogeneity

$$\text{for any lights } a \text{ and } b, \text{ and } t > 0, \text{ if } \sigma(a) = \sigma(b), \text{ then } \sigma(ta) = \sigma(tb). \tag{5}$$

Since $a \sim_\sigma b \Leftrightarrow \sigma(a) = \sigma(b)$, then the law of translatory invariance (4) entails the additivity law for $\sim_\sigma$ (i.e., for any a, b, and c in A if $a \sim_\sigma b$ then $(a+c) \sim_\sigma (b+c)$), and the law of positive homogeneity (5) entails the proportionality law (i.e., for any a, b in A and any positive real number t, if $a \sim_\sigma b$ then $ta \sim_\sigma tb$).

The law of translatory invariance (4) means that if the set of the lights matching light a is equal to that of b, then the set of the lights matching light $a + c$ will be equal to that of $b + c$. More specifically, if light a can substitute light b without breaking matching relation with an arbitrary light, then so will the mixture of light a with any light c for the mixture of lights b and c.

The law of positive homogeneity (5) implies that if the light a can substitute the light b without breaking matching relation with an arbitrary light, then it can also be done when the intensity of the lights a and b changes by a factor t. While these axioms are to be tested experimentally, they do not seem to contradict any known fact concerning colour matching (at least to the author's knowledge).

A serious consequence of the invalidity of the additivity and transitivity laws for colour matching is that the cone fundamentals cannot be derived from the colour matching functions. Indeed, if these laws fail then the cone fundamentals are not a linear transformation of the colour matching functions. Besides, if the colour match between lights a and b does not imply the equality of the colour mechanisms' responses to these lights, then there should be some volume around any point in the cone excitation space such that any other light the colour signal of which is inside this volume, will be a match. There is an indication that these volumes are not symmetrical (Logvinenko, Tyurin, & Sawey, 2002). Therefore, averaging matching data brings about a biased estimate of the volume center. This, in turn, will affect the estimate of the cone fundamentals.

The spectral sensitivity of the hypothetical colour mechanisms underlying intransitive colour matching, can be evaluated by using psychophysical techniques similar to those developed for measuring spatial and temporal sensitivity of visual channels (Logvinenko, 1995, 2003).

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